Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 3

Exercise 1 *(4 Points)*
Let $X, Y \in \mathcal{L}^2$. Prove the following assertions

(a) $\operatorname{var}(X) \geq 0$ and $\operatorname{var}(X) = 0$ if and only if X is a.s. constant.

(b) $\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y)$.

Exercise 2 *(4 Points)*
Let $X = (X_1, \ldots, X_d) \in \mathcal{L}^2(\Omega; \mathbb{R}^d)$. Define $\Sigma := (\sigma_{ij})_{i,j=1,\ldots,d}$ via

$$\sigma_{ij} = \operatorname{cov}(X_i, X_j).$$

Prove that Σ is symmetric and that

$$\sum_{i,j=1}^{d} \overline{\lambda}_i \lambda_j \sigma_{ij} = \mathbb{E}\left(|X^\lambda - \mathbb{E}(X^\lambda)|^2\right)$$

for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ where $X^\lambda := (\lambda X_1, \ldots, \lambda X_d)$. Conclude that Σ has only non-negative eigenvalues.

Exercise 3 *(4 Points)*
Let X and X_n be random variables with values in $\mathbb{R}$..

(a) Suppose that $X_n \longrightarrow X$ in probability. Prove that $f(X_n) \longrightarrow f(X)$ in probability for any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(b) Give an counterexample which shows that assertion (a) is not true if f is not continuous.

Hint: (a) Suppose first that X is bounded. Then use that f is uniformly continuous on compacts to deduce the assertion. Finally prove the assertion in the case where X is not necessarily bounded.
(b) Take $f(x) = 1_0(x)$ and consider a suitable sequence $X_n \longrightarrow 0$ in probability.